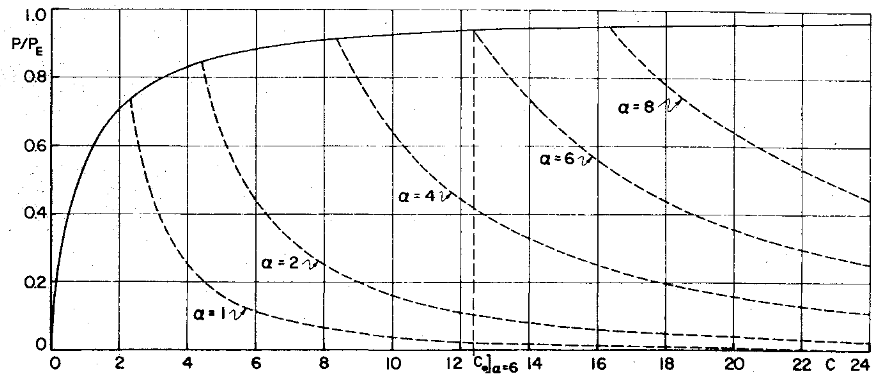

Fig. 2 Buckling load vs length.

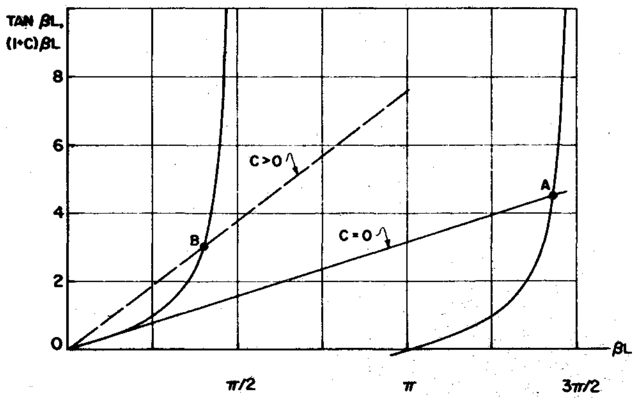

Fig. 3 Roots of Eq. (4).

Thus it is seen again that, for finite values of α, greater loads are required for instability for increasing lengths in the range $c < c_0$, where $c_0 = c_0(\alpha)$. The solid lines represent the instability caused by bending of member AB. For values $c_0 < c$, instability occurs due to buckling of member BC, and thus we arrive again at the classical case where a system of increasing length buckles under smaller loads.

The preceding results should be taken into account in the design of members conforming to a system represented by Fig. 1, where it is seen that increasing the value of c within the range $0 < c \leq c_0$ actually may improve design in the context of stability criteria.

References

[1] Timoshensko, S. and Gere, J., *Theory of Elastic Stability*, McGraw-Hill, New York, 1961.

Technical Comments

Comment on "Effective Thermal Property Improves Phase Change Paint Data"

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DRUMMOND et al.[1] have found "effective" steady-state thermophysical parameters for materials specimens representative of thin-wing models based on numerical analysis employing experimentally determined thermophysical material properties. For homogeneous materials, their data indicate that a single "effective" parameter that is a function of surface temperature only and is independent of heating rate describes the results; For nonhomogeneous samples the "effective" parameter depends both on surface temperature and on heating rate. In either case, the "effective" parameter values are different from those that would be deduced from the temperature-dependent properties of the materials measured in steady-state heat conduction tests. The purpose of this Comment is to show that these results are consequences of the basic equations of transient heat conduction in solids and hence applicable to a much broader range of temperature, materials, and geometric configurations than represented in the numerical calculations.

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Consider the three-dimensional transient heat conduction equation in which the thermal conductivity k, and the heat capacity c, are functions of temperature T:

$$\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(T)\frac{\partial T}{\partial y}\right] + \frac{\partial}{\partial z}\left[k(T)\frac{\partial T}{\partial z}\right] = \rho c(T)\frac{\partial T}{\partial t} \tag{1}$$

where ρ is the density.

The boundary condition for surface heat transfer given over the surface is

$$-k(T_s)\left(\frac{\partial T}{\partial n}\right)_s = Q \tag{2}$$

where s denotes surface, $\partial T/\partial n$ is the normal derivative, and Q is the heating rate, constant over the surface.

The boundary condition (2) and differential equation (1) may be normalized with respect to heating rate by dividing through by Q and Q^2, respectively. Making the substitutions

$$\xi = Qx \qquad \eta = Qy \qquad \zeta = Qz \qquad \tau = Q^2 t$$

leads to the equations

$$\frac{\partial}{\partial \xi}\left[k(T)\frac{\partial T}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[k(T)\frac{\partial T}{\partial \eta}\right] + \frac{\partial}{\partial \zeta}\left[k(T)\frac{\partial T}{\partial \zeta}\right] = \rho c(T)\frac{\partial T}{\partial \tau} \tag{3}$$

$$-k(T_s)\left(\frac{\partial T}{\partial \upsilon}\right)_s = I \quad (4)$$

where υ is the normal in transformed coordinates, ξ, η, ζ. This is a nonlinear problem, but the heating rate Q does not explicitly appear. The solution will be in the form $T=f(\xi, \eta, \zeta, \tau)$. Thus the temperature attained at the surface, with given material properties $k(T)$ and $c(T)$, will be the same for any heating rates Q whenever $\tau = Q^2 t$ is the same.

Drummond et al.[1] correlated their numerical results on the basis of the one-dimensional heat conduction equation of a semi-infinite solid with constant thermophysical properties

$$T_s - T_o = \frac{2t^{1/2} Q}{(k\rho c)^{1/2}_{\text{eff}} \pi^{1/2}} \quad (5)$$

where T_o was the initial temperature. For a given temperature rise T, $\tau = Q^2 t$ is constant according to the analysis given above, so that $[t(T)]^{1/2} = [\tau(T)]^{1/2}/Q$

Thus

$$T_s - T_o = \frac{2[\tau(T)]^{1/2}}{(kc\rho)^{1/2}_{\text{eff}} \pi^{1/2}}$$

or

$$(k\rho c)^{1/2}_{\text{eff}} = \frac{2[\tau(T)]^{1/2}}{(T_s - T_o)\pi^{1/2}} \quad (6)$$

The effect of temperature-dependent thermal conductivity $k(T)$ in the steady-state case may be dealt with through the Kirchoff transformation[2,3]

$$\theta = \int_{T_o}^{T} k(T_I)\, \mathrm{d}T_I \quad (7)$$

For the transient case, this transformation leads to

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{\rho c(T)}{k(T)} \frac{\partial \theta}{\partial t} \quad (8)$$

with the boundary condition

$$\left(\frac{\partial \theta}{\partial n}\right)_s = Q \quad (9)$$

θ is a single-valued function of T depending only on the material properties through Eq. (7). Although this form is sometimes useful for numerical computations, it is of special interest analytically only for the steady-state case, in which it leads to Laplace's equation and in the transient case when $\rho c(T)/k(T) = \text{constant}$ [i.e., $c(T) \propto k(T)$], in which case it leads to the usual linear equation of transient heat conduction with θ in place of T. Note that the preceding transformation, which normalizes the equation and boundary condition with respect to Q, may also be applied in this case.

A completely analogous analysis to that which led to Eqs. (3-6) can be carried through for the case of heat transfer from a fluid at the surface with the boundary condition

$$-k(T_s)\left(\frac{\partial T}{\partial n}\right)_s = h(T_{aw} - T_s) \quad (10)$$

where h is the heat-transfer coefficient and T_{aw} is the adiabatic wall temperature. In this case, Eq. (10) is divided through by h and the partial differential equation (1) is divided through by h^2. Analogously, the new time variable is $\tau = h^2 t$. Drummond et al. used the one-dimensional constant-property solution for this case, namely

$$[T_s(t) - T_s(o)] = [T_{aw} - T_s(o)][1 - e^{\gamma^2} \operatorname{erfc}\gamma] \quad (11)$$

where

$$\gamma = \frac{ht^{1/2}}{(k\rho c)^{1/2}_{\text{eff}}}$$

Again, according to the foregoing theory, a given value of T_s is attained when $\tau = h^2 t$ is constant regardless of the value of h. Therefore, in this case also, Eq. (11) leads to a value of $(k\rho c)^{1/2}_{\text{eff}}$ which is a function of temperature only and not of heat transfer rate h.

If k, ρ, and c are functions of the coordinates x, y, and z as in the second set of tests reported in Ref. 1, the situation is not so simple. However, the transformations described above can still be useful. The thermophysical constants must also be transformed such that (e.g., in the case of constant heating rate Q)

$$k(x, y, z, T) = k(\xi/Q, \eta/Q, \zeta/Q, T) \quad (12)$$

$$c(x, y, z, T) = c(\xi/Q, \eta/Q, \zeta/Q, T) \quad (13)$$

Thus, the results for different values of Q give the same temperature rises at equal values of $Q^2 t = \tau$ only when the thermophysical properties are varied with Q in accordance with the transformation of the spatial coordinates. Although this situation is more complicated than when properties do not vary spatially, the results may be used to reduce the number of parameters involved in calculating numerically or testing systematically over ranges of heat-transfer rates and thermophysical property variations, if standard test model configurations are to be used.

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[2] Jakob, M., *Heat Transfer*, Vol. I., Wiley, New York, 1949, pp. 102-103.

[3] Dusenberre, G.M., *Heat Transfer Calculations by Finite Differences*, International Textbook, Scranton, Pa., 1961, pp. 172-173.

Reply by Authors to A. H. Flax

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THE authors wish to thank Dr. Flax for calling attention to a generalization of their work.[1] His discussion regarding the independence of the effective thermophysical property on the surface heat-transfer rate should reduce the effort needed when analyzing homogeneous wind-tunnel model materials. Several points should be noted, however.

First, we never intended that our analysis be limited to only thin-wing sections of models. In fact, the method is normally applied to "thick" model regions, and, as discussed in Ref. 2, more care must be taken when examining "thin" sections by

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